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CONTENTS WITH ABSTRACT

Abstract: We extend an example due to Cutler-Missel (Comm. Alg., 1984) of a separable efi $p^{\omega+2}$-projective abelian p-group which is not C-decomposable and thick by showing that there exists even an inseparable efi $p^{\omega+3}$-projective p-torsion Σ-group which is neither C-decomposable nor thick. This supplies two recent results of ours in (Comm. Alg., 2008).

Abstract: In this study, we consider generalized recurrent, generalized Ricci-recurrent and generalized concircular recurrent contact metric manifolds, ξ belonging to k-nullity distribution. We show that there exist no generalized recurrent and generalized Ricci-recurrent contact metric manifolds, ξ belonging to k-nullity distribution, unless $k\alpha + \beta$ is everywhere zero. Furthermore, we find the characterizations of scalar curvatures of generalized recurrent and concircular recurrent manifolds of such type.

Abstract: We investigate the GH_k integral for functions defined on (possibly) unbounded subintervals of the extended real line and with values in Riesz spaces. Some convergence theorems are proved, together with a version of the Fundamental Formula of Calculus.

Abstract: Most of the matter in the universe can, in some form or other, be treated as a fluid and in several phenomena such as supernova explosions, jets in extragalactic radio sources, accretions onto neutron stars and black holes, high-energy particle beams, high-energy nuclear collisions etc. undergoes the relativistic motion. In the general relativity the matter content of the spacetime is described by the energy-momentum tensor which is determined from physical considerations dealing with the distribution of the matter and energy. Since the matter content of the universe is assumed to behave like a perfect fluid in the standard cosmological model, the physical motivation for studying Lorentzian manifolds is the assumption that a gravitational field may be effectively modelled by some Lorentzian metric defined on a suitable four dimensional manifold which is called the general relativistic spacetime. The object of the present paper is to study a type of a general relativistic spacetime with vanishing and as well as the divergence free W_2-curvature tensor.

Abstract: The aim of this paper is to combine the method of reflective function and the perturbation method. The set of a linear differential systems, the reflecting matrix for which is represented by a product of three exponential matrixes is allocated. It has allowed to obtain the sufficient conditions of existence of a family of periodic solutions close to a given solution of multidimensional nonlinear differential systems. Obtained results are illustrated by examples.

Abstract: In this paper, we give some criteria for the nonlinear differential systems to be simple systems and find out their reflective functions. The results are applied to the discussion of the behavior of solutions of these nonlinear differential systems. In particular, we discuss the qualitative behavior of solutions of Riccati equation.

Abstract: Based on (A, η)-monotonicity, a new class of nonlinear variational inclusion problems is presented. Since (A, η)-monotonicity generalizes A-monotonicity and H-monotonicity and in turn, generalizes maximal monotonicity, results thus obtained are general in nature and encompass a broad range of previous results.

Abstract: We use a structure from set theory to define a locally convex topology on ℓ_∞ which is coarser than the usual norm topology and gives rise to a smaller dual space. The two topologies have the same boundedness, compactness and weak compactness characteristics; under the new topology ℓ_∞ is complete but lacks most weak barrelledness properties. We also identify our space as a space of continuous functions on a certain pseudocompact, locally compact space and show that its topology is a Mackey topology.

Abstract: In the present paper, K denotes a complete, non-trivially valued, non-archimedean field. Entries of sequences, series and infinite matrices are in K. The purpose of the present paper is to extend Theorem 1 of [2] for double series, introduce the concept of convolution for double infinite matrices and to prove some basic results related to that concept in non-archimedean fields.

Abstract: In this paper, we study the classification of Stein manifolds equipped with codimension one (holomorphic) actions of $\mathbb{C}^n$. We regard the case of algebraic foliations on $\mathbb{C}^{n+1}$ and prove a linearization result. The other main result of this paper

Riemann-Stieltjes type integral operators

$$T_g f(z) = \int_0^1 f(tz)\Re g(tz)\frac{dt}{t} \quad \text{and} \quad L_g f(z) = \int_0^1 \Re f(tz)g(tz)\frac{dt}{t},$$

$z \in B$, between different mixed norm spaces of holomorphic functions $H_{p_1,q_1,\gamma_1}(B)$ and $H_{p_2,q_2,\gamma_2}(B)$.

Abstract: We present in this paper a differential version of Mirzakhani's recursion relation for the Weil-Petersson volumes of the moduli spaces of bordered Riemann surfaces. We discover that the differential relation, which is equivalent to the original integral formula of Mirzakhani, is a Virasoro constraint condition on a generating function for these volumes. We also show that the generating function for ψ and κ_1 intersections on $\overline{M}_{g,n}$ is a 1-parameter solution to the KdV hierarchy. It recovers the Witten-Kontsevich generating function when the parameter is set to be 0.

Abstract: In this paper, we prove some fixed point theorem for mappings satisfying contractive condition of integral type on general topological spaces using a τ-distance which is given by Aamri and El Moutawakil in [1]. Our results extend and generalize the results of Aamri and El Moutawakil [1], Branciari [4] and some others.

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COINCIDENCES AND FIXED POINTS OF NON-CONTINUOUS MAPS 229-237

Abstract: The main purpose of this paper is to obtain coincidence and common fixed point theorems for non-continuous maps using (IT)-commutativity. Some recent results are improved considerably.

REFLECTING FUNCTION AND PERIODIC SOLUTIONS OF DIFFERENTIAL SYSTEMS WITH SMALL PARAMETER*

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The aim of this paper is to combine the method of reflective function and the perturbation method. The set of a linear differential systems, the reflecting matrix for which is represented by a product of three exponential matrixes is allocated. It has allowed to obtain the sufficient conditions of existence of a family of periodic solutions close to a given solution of multidimensional nonlinear differential systems. Obtained results are illustrated by examples.

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1. Introduction

As we know, most of differential systems cannot be integrated in quadratures. Even so, some differential systems can be investigated on the qualitative level using the reflecting function introduced in [4].

In the present paper with the help of reflecting function and the small parameter method we research qualitative behaviour of solutions of multidimensional differential systems (see also [5]).

Reflecting Function. General Case

Consider the system

$$\dot{x} = X(t, x), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^n, \tag{1}$$

with a continuously differentiable right-hand side and with a general solution $\varphi(t; t_0, x_0)$.

For each such system, the *reflecting function* (RF) is defined (see [4, 5]) as $F(t, x) := \varphi(-t; t, x)$.

If system (1) is 2ω-periodic with respect to t, and F is its RF, then $F(-\omega, x) = \varphi(\omega; -\omega, x)$ is the Poincaré mapping of this system over the period $[-\omega,\, \omega]$.

A function $F(t, x)$ is a reflecting function of system (1) if and only if it is a solution of the system of partial differential equations (called a *basic relation*, BR)

$$\frac{\partial F(t,x)}{\partial t} + \frac{\partial F(t,x)}{\partial x} X(t,x) + X(-t, F(t,x)) = 0$$

. with the initial condition $F(0, x) \equiv x$.

Each continuously differentiable function F that satisfies the condition

$$F(-t, F(t, x)) \equiv F(0, x) \equiv x,$$

is a RF of the whole class of systems of the form (see [6])

$$\dot{x} = -\frac{1}{2}\frac{\partial F}{\partial x}\big(-t, F(t,x)\big)\left(\frac{\partial F(t,x)}{\partial t} - 2S(t,x)\right) - S\big(-t, F(t,x)\big), \tag{2}$$

where S is an arbitrary vector function such that solutions of the system (2) are uniquely determined by their initial conditions.

Therefore, all systems of the form (1) are split into equivalence classes of the form (2) so that each class is specified by a certain reflecting function referred to as the *RF of the class.*

For all systems of one class, the shift operator [3, pp. 11-13] on the interval $[-\omega, \omega]$ is the same. Therefore, all equivalent 2ω-periodic systems have a common mapping over the period, and the behaviors of the periodic solutions of these systems are the same.

Linear Case. Reflecting Matrix

Let system (1) be linear, i.e.

$$\dot{x} = P(t)x, \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^n, \tag{3}$$

and $\Phi(t)$ is its fundamental matrix of solutions.

Then general solution of system (3) is $\varphi(t; t_0, x_0) \equiv \Phi(t)\Phi^{-1}(t_0)x_0$. Therefore RF of system (3) is linear and $F(t,x) \equiv F(t)x$, where $F(t) := \Phi(-t)\Phi^{-1}(t)$. This matrix $F(t)$ is referred to as a *reflecting matrix* (RM) of system (3).

RM of any system satisfies the relation $F(-t)F(t) \equiv F(0) = E$, where E is the $n \times n$ unit matrix.

Differentiable matrix $F(t)$ is a RM of system (3) if and only if it is a solution of the system (basic relation)

$$\dot{F}(t) + F(t)P(t) + P(-t)F(t) = 0.$$

with the initial condition $F(0) = E$.

Any linear system with reflecting function $F(t)$ can be reduced in the form

$$\dot{x} = \left(-\frac{1}{2}F(-t)\dot{F}(t) + F(-t)R(t) - R(-t)F(t) \right) x,$$

where $R(t)$ is an arbitrary continuous real $n \times n$ matrix.

If matrix $P(t)$ is 2ω-periodic, and $F(t)$ is RM of system (3), then solutions μ_i, $i = \overline{1, n}$ of the equation $\det(F(-\omega) - \mu E) = 0$ are multiplicators of system (3).

See articles [5, 7-17], in which RF was also used for investigations of qualitative behaviour of solutions of differential systems.

2. Linear Systems with given Structure of the Reflecting Matrix

Consider the linear differential system

$$\dot{x} = P(t)x, \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^n, \tag{4}$$

where $P(t)$ is a twice continuously differentiable $n \times n$ matrix. In some cases (as it takes place for a periodic systems) the fundamental matrix $X(t)$ of the system (4) can be represented in the form

$$X(t) \equiv \Phi(t)\mathrm{e}^{-\frac{B}{2}t},$$

where $\Phi(t)$ is a continuous periodic $n \times n$ matrix; B is a constant $n \times n$ matrix. RM of such systems is

$$F(t) \equiv X(-t)X^{-1}(t) \equiv \Phi(-t)\mathrm{e}^{Bt}\Phi^{-1}(t).$$

With this in mind, we suppose what RM of system (4) is given by

$$F(t) \equiv \mathrm{e}^{At}\mathrm{e}^{Bt}\mathrm{e}^{At},$$

where A and B are constant $n \times n$ matrices.

Lemma 2.1. *Let RM of the system* (4) *be* $F(t) \equiv \mathrm{e}^{At}\mathrm{e}^{Bt}\mathrm{e}^{At}$, *where* A *and* B *are constant* $n \times n$ *matrixes. Then* $B = -2\,(A + P(0))$ *and*

$$2\left(P^2(0)A - 2P(0)AP(0) + AP^2(0)\right) - \left(A^2P(0) - 2AP(0)A + P(0)A^2\right)$$
$$+2\left(\dot{P}(0)P(0) - P(0)\dot{P}(0)\right) + \ddot{P}(0) = 0. \tag{5}$$

Proof. Writing out BR for the considered RM, we obtain the identity

$$F(t)\,(A + P(t)) + (A + P(-t))\,F(t) + \mathrm{e}^{At}B\mathrm{e}^{Bt}\mathrm{e}^{At} \equiv 0.$$

By setting $t = 0$, we obtain matrix B. Twice differentiating obtained identity and putting $t = 0$, we get (5).

Theorem 2.1. *Let*

$$P(t) \equiv \mathrm{e}^{-At}\mathrm{e}^{(A+P(0))t}S(t)\mathrm{e}^{-(A+P(0))t}\mathrm{e}^{At} + \mathrm{e}^{-At}P(0)\mathrm{e}^{At}, \tag{6}$$

where $S(t)$ *is an arbitrary odd continuous* $n \times n$ *matrix,* A *is constant* $n \times n$ *matrix. Then RM of the system* (4) *is* $F(t) \equiv \mathrm{e}^{At}\mathrm{e}^{-2(A+P(0))t}\mathrm{e}^{At}$.

And backwards. Let RM of the system (4) be $F(t) \equiv \mathrm{e}^{At}\mathrm{e}^{-2(A+P(0))t}\mathrm{e}^{At}$, where A is constant $n \times n$ matrix. Then exist odd $n \times n$ matrix $S(t)$ for which matrix $P(t)$ of the system (4) has the form (6).

Proof. Let matrix of the system (4) has the form (6). By checkout of the BR it is proved, that matrix $F(t) \equiv \mathrm{e}^{At}\mathrm{e}^{-2(A+P(0))t}\mathrm{e}^{At}$ is RM of the system (4).

Backwards. Let matrix $F(t) \equiv \mathrm{e}^{At}\mathrm{e}^{-2(A+P(0))t}\mathrm{e}^{At}$ be RM of the system (4). Then from BR we obtain the identity

$$F(t)\,(A+P(t)) - \mathrm{e}^{At}\mathrm{e}^{-2(A+P(0))t}\,(A+P(0))\,\mathrm{e}^{At} \equiv \\ -\,(A+P(-t))\,F(t) + \mathrm{e}^{At}\,(A+P(0))\,\mathrm{e}^{-2(A+P(0))t}\mathrm{e}^{At},$$

i.e.

$$F(t)\,(A+P(t)) - F(t)A - \mathrm{e}^{At}\mathrm{e}^{-2(A+P(0))t}P(0)\mathrm{e}^{At} \equiv \\ -\,(A+P(-t))\,F(t) + AF(t) + \mathrm{e}^{At}P(0)\mathrm{e}^{-2(A+P(0))t}\mathrm{e}^{At}.$$

Then we have

$$F(t)P(t) - \mathrm{e}^{At}\mathrm{e}^{-2(A+P(0))t}P(0)\mathrm{e}^{At} \equiv -P(-t)F(t) + \mathrm{e}^{At}P(0)\mathrm{e}^{-2(A+P(0))t}\mathrm{e}^{At}.$$

Premultiplying and postmultiplying the last identity by e^{-At}, we obtain

$$\mathrm{e}^{-2(A+P(0))t}\mathrm{e}^{At}P(t)\mathrm{e}^{-At} - \mathrm{e}^{-2(A+P(0))t}P(0) \equiv \\ -\,\mathrm{e}^{-At}P(-t)\mathrm{e}^{At}\mathrm{e}^{-2(A+P(0))t} + P(0)\mathrm{e}^{-2(A+P(0))t}.$$

Premultiplying and postmultiplying the above obtained identity by $\mathrm{e}^{A+P(0)}$, we have

$$\mathrm{e}^{-(A+P(0))t}\mathrm{e}^{At}P(t)\mathrm{e}^{-At}\mathrm{e}^{(A+P(0))t} - \mathrm{e}^{-(A+P(0))t}P(0)\mathrm{e}^{(A+P(0))t} \equiv \\ -\,\mathrm{e}^{(A+P(0))t}\mathrm{e}^{-At}P(-t)\mathrm{e}^{At}\mathrm{e}^{-(A+P(0))t} + \mathrm{e}^{(A+P(0))t}P(0)\mathrm{e}^{-(A+P(0))t}.$$

Hence the matrix

$$S(t) :\equiv \mathrm{e}^{-(A+P(0))t}\mathrm{e}^{At}P(t)\mathrm{e}^{-At}\mathrm{e}^{(A+P(0))t} - \mathrm{e}^{-(A+P(0))t}P(0)\mathrm{e}^{(A+P(0))t}$$

is odd matrix. From the last identity one can express the matrix $P(t)$ in the form (6).

Theorem 2.2. *Let matrix $P(t)$ of the system* (4) *has the form* (6), *then*

1) *the mapping of the 2ω–periodic system* (4) *over the period* $[-\omega, \omega]$ *is*

$$F(-\omega, x) = \mathrm{e}^{-A\omega}\mathrm{e}^{2(A+P(0))\omega}\mathrm{e}^{-A\omega}x;$$

2) *solution* $x(t)$ *of the system* (4) *with initial condition* $x(-\omega) = x_0$ *is* 2ω*-periodic solution if and only if* $F(-\omega, x_0) = x_0$;
3) *for any solution* $x(t)$ *of the system*(4) *the vector-function*

$$Y(t) \equiv \mathrm{e}^{-(A+P(0))t}\mathrm{e}^{At}x(t)$$

is even with respect to t.

Proof. It follows from the Theorem 2.1 that matrix

$$F(t) \equiv \mathrm{e}^{At}\mathrm{e}^{-2(A+P(0))t}\mathrm{e}^{At}$$

is RM of the system (4). Therefore, the assertions 1) and 2) of the theorem follow from [4, p. 11].

Since the matrix $F(t)$ is RM of the system (4), so for any solution $x(t)$ of the system (4) the identity

$$x(-t) \equiv \mathrm{e}^{At}\mathrm{e}^{-2(A+P(0))t}\mathrm{e}^{At}x(t)$$

is true. Premultiplying the last identity by $\mathrm{e}^{(A+P(0))t}\mathrm{e}^{-At}$, we obtain

$$\mathrm{e}^{(A+P(0))t}\mathrm{e}^{-At}x(-t) \equiv \mathrm{e}^{-(A+P(0))t}\mathrm{e}^{At}x(t).$$

Hence the function

$$Y(t) :\equiv \mathrm{e}^{-(A+P(0))t}\mathrm{e}^{At}x(t)$$

is even function.

The following assertion is a consequence of the Theorem 2.1 for $S(t) \equiv \gamma(t)E$.

Lemma 2.2. *Let matrix of the system* (4) *be*

$$P(t) \equiv \mathrm{e}^{-At}P(0)\mathrm{e}^{At} + \gamma(t)E, \tag{7}$$

where A *is a constant* $n \times n$ *matrix, and* $\gamma(t)$ *is a continuous scalar odd function. Then RM of the system* (4) *is* $F(t) \equiv \mathrm{e}^{At}\mathrm{e}^{-2(A+P(0))t}\mathrm{e}^{At}$.

Remark 2.1. If matrix of the system (4) has the form

$$P(t) \equiv \mathrm{e}^{-At}P(0)\mathrm{e}^{At}, \tag{8}$$

then Lemma 2.2 is valid.

In some case we can think that matrix of the system (4) is an approximation to Fourier series. With this in mind we consider the system (4) with matrix

$$P(t) \equiv A_1 + B_1 \cos mt + C_1 \sin mt + B_2 \cos rt + C_2 \sin rt, \tag{9}$$

where A_1, B_1, B_2, C_1, C_2 are constant $n \times n$ matrixes, and $m, r \in \mathbb{R}$.

Theorem 2.3. *Let matrix of the system* (4) *has the form* (9) *and* $m \neq 0$. *Then matrix of the system* (4) *has the form* (8) *if and only if*

$$\begin{cases} A_1 A = A A_1, \\ B_1 A - A B_1 = m C_1, \\ A C_1 - C_1 A = m B_1, \\ B_2 A - A B_2 = r C_2, \\ A C_2 - C_2 A = r B_2. \end{cases} \tag{10}$$

Proof. *Necessity.* Let matrix (9) of the system (4) has the form (8). By differentiating the identity (8), we obtain

$$\dot{P}(t) \equiv \mathrm{e}^{-At} \left(P(0)A - AP(0)\right) \mathrm{e}^{At}.$$

Using identities (8), we have $\dot{P}(t) \equiv P(t)A - AP(t)$. Applying $k-1$ times differentiation to latter identity, we obtain

$$\frac{\mathrm{d}^k P(t)}{\mathrm{d}t^k} \equiv \frac{\mathrm{d}^{k-1} P(t)}{\mathrm{d}t^{k-1}} A - A \frac{\mathrm{d}^{k-1} P(t)}{\mathrm{d}t^{k-1}}, \quad \forall k \in \mathbb{N}.$$

We make replacement of the variable $\tau = mt$, then for any $k \in \mathbb{N}$ we have

$$\frac{\mathrm{d}^k P\left(\frac{\tau}{m}\right)}{\mathrm{d}\left(\frac{\tau}{m}\right)^k} \equiv \frac{\mathrm{d}^{k-1} P\left(\frac{\tau}{m}\right)}{\mathrm{d}\left(\frac{\tau}{m}\right)^{k-1}} A - A \frac{\mathrm{d}^{k-1} P\left(\frac{\tau}{m}\right)}{\mathrm{d}\left(\frac{\tau}{m}\right)^{k-1}},$$

i.e.

$$m \frac{\mathrm{d}^k P\left(\frac{\tau}{m}\right)}{\mathrm{d}\tau^k} \equiv \frac{\mathrm{d}^{k-1} P\left(\frac{\tau}{m}\right)}{\mathrm{d}\tau^{k-1}} A - A \frac{\mathrm{d}^{k-1} P\left(\frac{\tau}{m}\right)}{\mathrm{d}\tau^{k-1}}.$$

From last identities by setting $\tau = 0$, we obtain equalities

$$(A_1 + B_1 + B_2) A - A (A_1 + B_1 + B_2) = m \left(C_1 + \frac{r}{m} C_2\right), \tag{11a}$$

$$\left(C_1 + \frac{r}{m} C_2\right) A - A \left(C_1 + \frac{r}{m} C_2\right) = -m \left(B_1 + \frac{r^2}{m^2} B_2\right), \tag{11b}$$

$$-\left(B_1 + \frac{r^2}{m^2} B_2\right) A + A \left(B_1 + \frac{r^2}{m^2} B_2\right) = -m \left(C_1 + \frac{r^3}{m^3} C_2\right), \tag{11c}$$

$$A\left(C_1+\frac{r^{2k-1}}{m^{2k-1}}C_2\right)-\left(C_1+\frac{r^{2k-1}}{m^{2k-1}}C_2\right)A=m\left(B_1+\frac{r^{2k}}{m^{2k}}B_2\right), \quad (11d)$$

$$\left(B_1+\frac{r^{2k}}{m^{2k}}B_2\right)A-A\left(B_1+\frac{r^{2k}}{m^{2k}}B_2\right)=m\left(C_1+\frac{r^{2k+1}}{m^{2k+1}}C_2\right). \quad (11e)$$

From (11d) it follows, that for any $k\in\mathbb{N}$ and $s=k+1,k+2,\dots$ we have relation

$$A\left(C_1+\frac{r^{2s-1}}{m^{2s-1}}C_2\right)-\left(C_1+\frac{r^{2s-1}}{m^{2s-1}}C_2\right)A=$$
$$A\left(C_1+\frac{r^{2k-1}}{m^{2k-1}}C_2\right)-\left(C_1+\frac{r^{2k-1}}{m^{2k-1}}C_2\right)A+$$
$$\left(\frac{r^{2s-1}}{m^{2s-1}}-\frac{r^{2k-1}}{m^{2k-1}}\right)(AC_2-C_2A)=m\left(B_1+\frac{r^{2s}}{m^{2s}}B_2\right),$$

i.e.

$$\left(\frac{r^{2s-1}}{m^{2s-1}}-\frac{r^{2k-1}}{m^{2k-1}}\right)(AC_2-C_2A)=$$
$$m\left(B_1+\frac{r^{2s}}{m^{2s}}B_2\right)-m\left(B_1+\frac{r^{2k}}{m^{2k}}B_2\right).$$

Whence we obtain $AC_2-C_2A=rB_2$. Analogously from (11e) follows $B_2A-AB_2=rC_2$. Using the obtained equalities, from (11d) and (11e) accordingly, we obtain $AC_1-C_1A=mB_1$, $B_1A-AB_1=mC_1$. From (11a) we have $A_1A=AA_1$. Thus equalities (10) are valid. We remark, that systems (10) and (11) are equivalent.

Sufficiency. Let matrix of the system (4) has the form (9) and equalities (10) are valid. We check, that for any $k\in\mathbb{N}$ the identity

$$\frac{\mathrm{d}^kP}{\mathrm{d}t^k}(0)\equiv\frac{\mathrm{d}^{k-1}P}{\mathrm{d}t^{k-1}}(0)A-A\frac{\mathrm{d}^{k-1}P}{\mathrm{d}t^{k-1}}(0)$$

is valid. Let $Q(t):=\mathrm{e}^{-At}P(0)\mathrm{e}^{At}$. As it is proved above, for any $k\in\mathbb{N}$ identity

$$\frac{\mathrm{d}^kQ(t)}{\mathrm{d}t^k}\equiv\frac{\mathrm{d}^{k-1}Q(t)}{\mathrm{d}t^{k-1}}A-A\frac{\mathrm{d}^{k-1}Q(t)}{\mathrm{d}t^{k-1}}$$

is valid. As $Q(0)=P(0)$ then

$$\frac{\mathrm{d}^kQ}{\mathrm{d}t^k}(0)=\frac{\mathrm{d}^kP}{\mathrm{d}t^k}(0).$$

Functions $P(t)$ and $Q(t)$ are analytical, hence $Q(t) \equiv P(t)$.

3. Systems with Small Parameter

Obtained results for linear differential system can be extended for nonlinear systems with small parameter.

Consider the nonlinear differential system depending on parameter ν

$$\dot{x} = f(t, x, \nu), \quad t \in \mathbb{R}, \quad x \in D \subset \mathbb{R}^n, \tag{12}$$

where f is a continuous 2ω-periodic vector function for all t, small $|\nu|$, and also continuously differentiable with respect to components of a vector x. Let $x = g_0(t)$ be a 2ω-periodic solution of the system (12) in which $\nu = 0$.

Using concept of a reflecting matrix we can reformulate a following three theorems.

Theorem 3.1. *Let matrix*

$$F(t) \equiv \mathrm{e}^{At}\mathrm{e}^{-2(A+P(0))t}\mathrm{e}^{At}$$

be the RM of the linear system (3) *with matrix*

$$P(t) = \frac{\partial f}{\partial x}\big(t, g_0(t), 0\big).$$

If there is no unit among solutions μ_i of the equation

$$\det\left(\mathrm{e}^{-A\omega}\mathrm{e}^{2(A+P(0))\omega}\mathrm{e}^{-A\omega} - \mu E\right) = 0,$$

then system (12) *with sufficiently small* $|\nu|$ *has the unique* 2ω*-periodic solution* $x = x(t, \nu)$ *with an initial point* $x(0, \nu)$ *close to* $g_0(0)$. *Besides,* $x(t, \nu)$ *is a continuous function with respect to* (t, ν), *and* $x(t, 0) = g_0(t)$. *If, moreover,* f *is continuously differentiable with respect to* ν, *then* $x(t, \nu)$ *is also continuously differentiable.*

Proof. Since multiplicators μ_i for 2ω–periodic linear system with the RM

$$F(t) \equiv \mathrm{e}^{At}\mathrm{e}^{-2(A+P(0))t}\mathrm{e}^{At}$$

are solutions of the equation

$$\det\left(\mathrm{e}^{-A\omega}\mathrm{e}^{2(A+P(0))\omega}\mathrm{e}^{-A\omega} - \mu E\right) = 0,$$

therefore validity of this theorem follows from the Theorem 2.3 in [2, p. 488].

Quasilinear Systems

Let, in particular, the system (12) be quasilinear 2ω-periodic system

$$\dot{x} = P(t)x + f(t) + \nu\varphi(t, x), \quad t \in \mathbb{R},\ x \in D \subset \mathbb{R}^n, \tag{13}$$

where $P(t)$ is a continuous 2ω-periodic $n \times n$ matrix; $f(t)$ and $\varphi(t, x)$ are continuous 2ω–periodic vector function with respect to t, and also $\varphi(t, x)$ is continuously differentiable with respect to components of a vector x; ν is a small parameter. The following assertion is a consequence of the Theorem 3.1.

Theorem 3.2. *Let matrix*

$$F(t) \equiv \mathrm{e}^{At}\mathrm{e}^{-2(A+P(0))t}\mathrm{e}^{At}$$

be the RM of the system $\dot{x} = P(t)x$. If there is no unit among solutions μ_i of the equation

$$\det\left(\mathrm{e}^{-A\omega}\mathrm{e}^{2(A+P(0))\omega}\mathrm{e}^{-A\omega} - \mu E\right) = 0,$$

then system (13) *with sufficiently small $|\nu|$ has the unique 2ω-periodic solution $x = x(t, \nu)$ which satisfies the condition*

$$\lim_{\nu \to 0} x(t, \nu) = x_0(t),$$

where $x_0(t)$ is a 2ω-periodic solution of the system $\dot{x} = P(t)x + f(t)$.

Proof. Having observed that multiplicators μ_i for 2ω-periodic linear system with the RM

$$F(t) \equiv \mathrm{e}^{At}\mathrm{e}^{-2(A+P(0))t}\mathrm{e}^{At}$$

are solutions of the equation

$$\det\left(\mathrm{e}^{-A\omega}\mathrm{e}^{2(A+P(0))\omega}\mathrm{e}^{-A\omega} - \mu E\right) = 0,$$

therefore we obtain the assertion of the theorem from the Theorem in [1, p. 226].

Autonomous Systems

Now consider the autonomous differential system depending from parameter ν

$$\dot{x} = f(x, \nu), \quad x \in D \subset \mathbb{R}^n, \quad \nu \in \mathbb{R}, \tag{14}$$

where f is a continuous vector function with respect to small $|\nu|$ and $x \in D$, also continuously differentiable with respect to components of a vector x. Let $x = \eta(t) \not\equiv$ constant be a $2\omega_0$-periodic solution of the system $\dot{x} = f(x, 0)$.

Theorem 3.3. *Let matrix*

$$F(t) \equiv \mathrm{e}^{At}\mathrm{e}^{-2(A+P(0))t}\mathrm{e}^{At}$$

be the RM of the linear system (3) *with matrix*

$$P(t) = \frac{\partial f}{\partial x}\big(\eta(t), 0\big).$$

If among solutions μ_i of the equation

$$\det\Big(\mathrm{e}^{-A\omega_0}\mathrm{e}^{2(A+P(0))\omega_0}\mathrm{e}^{-A\omega_0} - \mu E\Big) = 0$$

there is unique simple unit, then system (14) *with sufficiently small $|\nu|$ has the unique periodic solution $x = x(t,\nu)$ close to $\eta(t)$ with period $\omega = \omega(\nu)$ close to $2\omega_0$. Moreover, $x(t,\nu)$ and $\omega(\nu)$ are continuous and $x(t,0) = \eta(t)$, $\omega(0) = 2\omega_0$.*

Proof. Since multiplicators μ_i for $2\omega_0$–periodic linear system with the RM

$$F(t) \equiv \mathrm{e}^{At}\mathrm{e}^{-2(A+P(0))t}\mathrm{e}^{At}$$

are solutions of the equation

$$\det\Big(\mathrm{e}^{-A\omega_0}\mathrm{e}^{2(A+P(0))\omega_0}\mathrm{e}^{-A\omega_0} - \mu E\Big) = 0,$$

therefore validity of this theorem follows from the Theorem 2.4 in [2, p. 488].

4. Some Examples

Example 4.1. Consider the quasilinear π–periodic system

$$\begin{cases} \dot{x} = \frac{1}{2}\big((1-\cos 2t)x - (2-\sin 2t)y\big) + f_1(t) + \nu\varphi_1(t,x,y), \\ \dot{y} = \frac{1}{2}\big((2+\sin 2t)x + (1+\cos 2t)y\big) + f_2(t) + \nu\varphi_2(t,x,y), \end{cases}$$

where $f_1(t)$, $f_2(t)$, $\varphi_1(t,x,y)$, $\varphi_2(t,x,y)$ are continuous π–periodic with respect to t scalar functions; moreover $\varphi_1(t,x,y)$, $\varphi_2(t,x,y)$ are continuously differentiable with respect to x and y; ν is a small parameter.

It follows from the Theorem 2.3 that respective linear homogeneous system has matrix of the form (8), where

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then, by Lemma 2.2, the RM of this system is

$$F(t) \equiv \frac{\mathrm{e}^{-t}}{30} \begin{pmatrix} F_1(t) & F_2(t) \\ F_3(t) & F_4(t) \end{pmatrix},$$

where

$$\begin{aligned} F_1(t) &\equiv \left(15 + 4\sqrt{15}\right) \cos(2 - \sqrt{15})t + \left(15 - 4\sqrt{15}\right) \cos(2 + \sqrt{15})t + \\ &\quad 2\sqrt{15} \sin \sqrt{15}t, \\ F_2(t) &\equiv -\left(15 + 4\sqrt{15}\right) \sin(2 - \sqrt{15})t - \left(15 - 4\sqrt{15}\right) \sin(2 + \sqrt{15})t, \\ F_3(t) &\equiv -F_2(t), \\ F_4(t) &\equiv F_1(-t). \end{aligned}$$

Therefore, multiplicators of this linear homogeneous system are

$$\mu_{1,2} = -\mathrm{e}^{\pi/2} \left(\cos \tfrac{\sqrt{15}\pi}{2} \pm i \sin \tfrac{\sqrt{15}\pi}{2}\right).$$

It follows from the Theorem 3.2 that the considered system has an unique π–periodic solution for sufficiently small $|\nu|$.

Example 4.2. Consider the system

$$\begin{cases} \dot{x} = y + x\left(1 - x^2 - y^2\right) + f_1(x, y, \nu), \\ \dot{y} = -x + y\left(1 - x^2 - y^2\right) + f_2(x, y, \nu), \end{cases}$$

where f_1, f_2 are continuous scalar functions for small $|\nu|$, and also continuously differentiable with respect to x and y; moreover,

$$f_i(x, y, 0) = \frac{\partial f_i}{\partial x}(\sin t, \cos t, 0) = \frac{\partial f_i}{\partial y}(\sin t, \cos t, 0) = 0, \quad i = 1, 2.$$

If $\nu = 0$ then $x_0 = \sin t$, $y_0 = \cos t$ is the 2π-periodic solution of considered system. The respective variational system for this solution is

$$\begin{cases} \dot{x} = (-1 + \cos 2t)\, x + (1 - \sin 2t)\, y, \\ \dot{y} = (-1 - \sin 2t)\, x + (-1 - \cos 2t)\, y. \end{cases}$$

It follows from the Theorem 2.3 that this system has matrix of the form (8), where

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then, by Lemma 2.2, the RM of this system is

$$F(t) \equiv \frac{1}{2}\begin{pmatrix} 1 - \mathrm{e}^{4t} + \left(1 + \mathrm{e}^{4t}\right)\cos 2t & -\left(1 + \mathrm{e}^{4t}\right)\sin 2t \\ \left(1 + \mathrm{e}^{4t}\right)\sin 2t & \mathrm{e}^{4t} - 1 + \left(1 + \mathrm{e}^{4t}\right)\cos 2t \end{pmatrix}.$$

Therefore, multiplicators of the variational system are $\mu_1 = 1$, $\mu_2 = \mathrm{e}^{-4\pi}$.

It follows from the Theorem 3.3 that the considered system with sufficiently small $|\nu|$ has the unique periodic solution $x = x(t,\nu)$, $y = y(t,\nu)$ close to $x_0(t)$, $y_0(t)$ with period $\omega = \omega(\nu)$ close to 2π. Moreover, this solution and its period are continuous, and $x(t,0) = \sin t$, $y(t,0) = \cos t$, $\omega(0) = 2\pi$.

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