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OPTIMALITY OF SYSTEMS OF IDENTICALLY DISTRIBUTED COMPETING PROCESSES

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This paper considers the problem of optimal distribution of computational resources of multiprocessor systems based on the principles of parallelization and conveyorization. The necessary conditions and criteria of efficiency and optimality of systems of identically distributed competing processes are obtained that take into account the overheads required in the context of their run time.

Keywords: *system of competing processes, distributed processing, optimality and efficiency criteria.*

Optimal distribution of computational resources of multiprocessor systems (MSs) based on the principles of parallelization and conveyorization is one of the most important problems in creating efficient system and application software [1, 2]. An important place in solving it is occupied by problems of optimal organization of competing processes using common program resources. The solution of these problems determines not only the efficiency of using MSs but also their capabilities to solve complicated problems from various areas of knowledge.

It is relevant to note that, in solving problems of optimal organization of competing processes, mathematical models based on lumped processing are most developed [2]. At the present time, MSs based on distributed processing have continued to play an increasingly important part [3–5]. In this connection, the problems of construction and investigation of mathematical models of optimal organization of competing processes during distributed processing are especially topical.

As in [5], a mathematical model of distributed processing of competing processes includes the following parameters: p, $p \geq 2$, is the numbers of processors of a multiprocessor system, n, $n \geq 2$, is the number of competing processes, s, $s \geq 2$, is the number of blocks Q_j of a structured program resource (PR), $j = \overline{1, s}$, and $[t_{ij}]$, $i = \overline{1, n}$, $j = \overline{1, s}$, is the matrix of run times of execution of blocks with the help competing processes. It is assumed that all n processes use one copy of the PR structured in the form of blocks.

Let us introduce the parameter $\varepsilon > 0$ that characterizes the time of additional system expenditures for the organization of parallel use of PR blocks by a set of distributed competing processes.

In this article, the necessary conditions and efficiency and optimality criteria of systems of identically distributed competing processes are considered, taking into account the overheads required depending on their run time.

1. RUN TIME OF DISTRIBUTED COMPETING PROCESSES

Definition 1. We call a system of competing processes identically distributed if the run times of execution of all PR blocks Q_j, $j = \overline{1, s}$, with the help of each ith process, $i = \overline{1, n}$, are equal to one another, i.e., we have

$$\begin{gathered} t_{11} = t_{12} = \ldots = t_{1s} = t'_1, \\ t_{21} = t_{22} = \ldots = t_{2s} = t'_2, \\ \ldots\ldots\ldots\ldots \\ t_{n1} = t_{n2} = \ldots = t_{ns} = t'_n, \end{gathered}$$

where $(t'_1, t'_2, \ldots, t'_n)$ are run times of execution of each block Q_j, $j = \overline{1, s}$, with the help of all n processes.

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Definition 2. We call an identically distributed system of competing processes stationary if we have $t'_1 = t'_2 = \ldots = t'_n = t$.

As is shown in [5], for an identically distributed system of processes competing in the asynchronous and second synchronous modes, the minimal total run time can be computed by the formulas

$$T_{\text{op}}^{\text{as}} = T_{\text{op}}^{2} = \begin{cases} T^n + (s-1)t_{\max}^n & \text{when } s \le p \text{ or } s > p \text{ but } T^n \le p t_{\max}^n; \\ kT^n + (p-1)t_{\max}^n & \text{when } s = kp,\ k > 1, \text{ and } T^n > p t_{\max}^n; \\ (k+1)T^n + (r-1)t_{\max}^n & \text{when } s = kp + r,\ k \ge 1,\ 1 \le r < p, \text{ and } T^n > p t_{\max}^n. \end{cases}$$

Here, $T^n = \sum_{i=1}^{n} t'_i$ is the total run time of execution of each block Q_j with the help of n processes and $t_{\max}^n = \max_{1 \le i \le n} t'_i$.

For the first synchronous mode that provides the continuous execution of PR blocks within each process, the minimal total run time of a given amount of computations in the MS being considered is specified by the relationships

$$T_{\text{op}}^{1} = \begin{cases} T^n + (s-1)\left[t'_n + \sum_{i=2}^{n} \max\{t'_{i-1} - t'_i, 0\} \right] & \text{when } s \le p; \\ kT_{\text{op}}^{1}(p, n, p) - (k-1)\min\{\omega_1, \omega_2\} & \text{when } s = kp,\ k > 1; \\ kT_{\text{op}}^{1}(p, n, p) + T_{\text{op}}^{1}(p, n, r) - (k-1)\min\{\omega_1, \omega_2\} - \min\{\xi_1, \xi_2\}, & \\ \qquad \text{when } s = kp + r,\ k \ge 1,\ 1 \le r < p. & \end{cases}$$

Here, we have

$$T_{\text{op}}^{1}(p, n, p) = T^n + (p-1)\left[t'_n + \sum_{i=2}^{n} \max\{t'_{i-1} - t'_i, 0\} \right], \tag{1}$$

$$\omega_1 = (p-1)\min\{t'_1, t'_n\}, \quad \omega_2 = T_{\text{op}}^{1}(p, n, p) - p \max_{1 \le i \le n} t'_i,$$

$$T_{\text{op}}^{1}(p, n, r) = T^n + (r-1)\left[t'_n + \sum_{i=2}^{n} \max\{t'_{i-1} - t'_i, 0\} \right], \tag{2}$$

$$\xi_1 = (r-1)\min\{t'_1, t'_n\} + (p-r)t'_n,$$

$$\xi_2 = T_{\text{op}}^{1}(p, n, p) - \max_{1 \le i \le n} \left(T_{\text{op}}^{1}(p, i, p) - T_{\text{op}}^{1}(p, i, r) + r t'_i \right).$$

In this case, $T_{\text{op}}^{1}(p, i, p)$ and $T_{\text{op}}^{1}(p, i, r)$ are found by formulas (1) and (2) after substitution of i for n.

Taking into account overheads $\varepsilon > 0$ when $s \le p$, the minimal total run time of systems of identically distributed competing processes in the asynchronous and second synchronous modes is determined by the formula

$$T(p, n, s, \varepsilon) = T_{\text{op}}^{\text{as}}(p, n, s, \varepsilon) = T_{\text{op}}^{2}(p, n, s, \varepsilon) = T_{\varepsilon}^{n} + (s-1)t_{\max}^{\varepsilon}, \tag{3}$$

where $T_{\varepsilon}^{n} = \sum_{i=1}^{n} t_i^{\varepsilon}$, $t_{\max}^{\varepsilon} = \max_{1 \le i \le n} t_i^{\varepsilon}$, and $t_i^{\varepsilon} = t'_i + \varepsilon$.

In the case of an identically distributed stationary system of competing processes, their minimal total run time for all the three basic modes is found, taking into account an additional system expenditures ε, by the formulas

$$\overline{T}_\varepsilon = \begin{cases} (n+s-1)t_\varepsilon, & p \ge \min\{n,s\}; \\ (kn+p-1)t_\varepsilon, & p < \min\{n,s\},\ \ s = kp,\ \ k > 1; \\ ((k+1)n + r - 1)t_\varepsilon, & p < \min\{n,s\},\ \ s = kp + r,\ \ k \ge 1,\ \ 1 \le r < p. \end{cases} \tag{4}$$

Here, $t_\varepsilon = T^n / n + \varepsilon$, $T^n = nt$. In formulas (3) and (4), the parameter ε is explicitly taken into account.

2. EFFICIENCY AND OPTIMALITY OF SYSTEMS OF COMPETING PROCESSES WHEN THE NUMBER OF PROCESSORS IS SUFFICIENT

Definition 3. An identically distributed system of competing processes with a fixed p and $s \ge 2$ is called efficient if the value of $\Delta_\varepsilon(n) = sT^n - T(p,n,s,\varepsilon) \ge 0$, where sT^n is the time of sequential execution of s blocks with the help of all n processes.

Definition 4. An identically distributed efficient system is called optimal if the value of $\Delta_\varepsilon(n)$ is maximum.

We will show that it suffices to search for the optimal identically distributed system among identically distributed efficient systems.

THEOREM 1. For any identically distributed efficient system of competing processes, there exists a more efficient identically distributed stationary system when $s \le p$ and $\varepsilon > 0$.

Proof. According to Definition 3, the condition of efficiency of an identically distributed system of competing processes is of the form

$$\Delta_\varepsilon(n) = (s-1)(T^n - t_{\max}^n) - (n+s-1)\varepsilon \ge 0 \tag{5}$$

or of the form

$$\overline{\Delta}_\varepsilon(n) = (s-1)(T^n - t) - (n+s-1)\varepsilon \ge 0, \text{ where } t = T^n / n, \tag{6}$$

when such an identically distributed system is stationary.

Let us show that the inequality $\overline{\Delta}_\varepsilon(n) > \Delta_\varepsilon(n)$ is true. To this end, we consider the difference $\overline{\Delta}_\varepsilon(n) - \Delta_\varepsilon(n) = (s-1)(t_{\max}^n - t)$. The case when $\overline{\Delta}_\varepsilon(n) - \Delta(n) \le 0$ takes place only if $t_{\max}^n \le t$. The latter inequality cannot be true since the fact of nonstationarity of an identically distributed system would imply $\sum_{i=1}^{n} t_i' < nt = T^n$ but, by the condition, we have $\sum_{i=1}^{n} t_i' = T^n$. We obtain the contradiction $T^n < T^n$, which proves the theorem.

THEOREM 2. In order that an identically distributed efficient system of competing processes be optimal when $s \le p$, it is necessary and sufficient that it be stationary.

Theorem 2 follows from Theorem 1 and Definition 4.

THEOREM 3. An identically distributed system with $n \ge 3$ competing processes in an MS with $p \ge 3$ processors is efficient when $n = s \ne 3$, $s \le p$, and $\varepsilon \le \min_{1 \le i \le n} t_i'$ if and only if the condition $sn \ge 2(n+s-1)$ is true.

The proof of the theorem is based on the analysis of the inequalities presented below. According to condition (5), the efficiency condition is equivalent to the inequality

$$\frac{T^n - t_{\max}^n}{\varepsilon} \ge \frac{n+s-1}{s-1}.$$

By virtue of the condition $\varepsilon \le \min_{1 \le i \le n} t_i'$ of the theorem, the following inequality holds true:

$$\frac{T^n - t_{\max}^n}{\varepsilon} \ge n-1.$$

Moreover, the condition $sn \ge 2(n+s-1)$ of the theorem is equivalent to the inequality

$$n-1 \ge \frac{n+s-1}{s-1}.$$

Let us formulate the necessary and sufficient conditions (criteria) of existence of an efficient system of identically distributed competing processes with a sufficient number of processors $s \le p$, depending on overheads $\varepsilon > 0$.

THEOREM 4. In order that an identically distributed efficient system of competing processes exist with given $p \ge 3$, T^n, and $\varepsilon > 0$ when $s \le p$, it is necessary and sufficient that the following conditions be true:

$$\varepsilon \le \begin{cases} \varphi(1+\sqrt{s}) & \text{if } \sqrt{s} \text{ is integer;} \\ \max\{\varphi(1+[\sqrt{s}]) \quad f(2+[\sqrt{s}])\} & \text{if } \sqrt{s} \text{ is not integer.} \end{cases}$$

Here, $\varphi(x) = \dfrac{(s-1)T^n(x-1)}{x(x+s-1)}$, $[x]$ is the greatest integer that does not exceed x.

Proof. The necessity follows from Definition 3 and Theorem 1. In fact, by virtue of formula (6), the efficiency condition is of the form

$$\overline{\Delta}_\varepsilon(n) = sT^n - \overline{T}_\varepsilon = (s-1)(T^n - t) - (n+s-1)\varepsilon \ge 0$$

that is equivalent to the following inequality:

$$\varepsilon \le \frac{(s-1)T^n(n-1)}{n(n+s-1)}. \tag{7}$$

Introducing the function $\varphi(x) = (s-1)T^n(x-1)/x(x+s-1)$, $x > 0$, we can easily show that it reaches its maximum when $x = 1+\sqrt{s}$.

Choosing $n = x = 1+\sqrt{s}$ if $\sqrt{s}$ is integer or $n = x \in \{1+[\sqrt{s}], 2+[\sqrt{s}]\}$ if $\sqrt{s}$ is not integer, we obtain an efficient system of identically distributed processes, which proves the necessity.

The sufficiency follows from inequality (7) since $\varphi(x)$ reaches its maximum when $n = x = 1+\sqrt{s}$ if $\sqrt{s}$ is integer or when $n = x \in \{1+[\sqrt{s}],\ 2+[\sqrt{s}]\}$ if $\sqrt{s}$ is not integer.

Comment. When $p = s = 2$, an identically distributed system of competing processes is efficient if the relation $\dfrac{\varepsilon}{T^n} \le \dfrac{n-1}{n(n+1)}$ is true.

In Fig. 1, the plot of the function $y = \varphi(x)$, $x > 0$, is presented for fixed $s = 50$, $T^n = 7$, and $\varepsilon = 5$. The existence of an identically distributed efficient system of competing processes is determined by the domain Ω. All the integer points of the segment Ω are the values of n for which the system is efficient; here, we have $x_1 = 1+[\sqrt{s}] = 8$, $x^* = 1+\sqrt{s}$, and $x_2 = 2+[\sqrt{s}] = 9$.

The solution of the problem of optimality of an identically distributed system that consists of n competing processes for a sufficient number of processors follows from the theorem formulated below.

THEOREM 5. In order that an identically distributed efficient system of competing processes be optimal with $s \le p$ and given $p \ge 2$, T^n, and $\varepsilon > 0$, it is necessary and sufficient that it be stationary and the number of processes n in the system be equal to the value of $\left[\sqrt{\dfrac{(s-1)T^n}{\varepsilon}}\right]$ or $\left[\sqrt{\dfrac{(s-1)T^n}{\varepsilon}}\right]+1$ for which the function $\overline{\Delta}_\varepsilon(x)$ assumes its maximum. Here, $[x]$ is the greatest integer that does not exceed x.

Proof. By virtue of Theorem 2, one should search for the optimal identically distributed system in the class of stationary systems. Then it is obvious that we have $\overline{\Delta}_\varepsilon(n) = (s-1)T^n(1-1/n) - (n+s-1)\varepsilon$.

Necessity. Let us introduce a function of a real argument x in the form

$$\overline{\Delta}_\varepsilon(x) = (s-1)T^n(1-1/x) - (x+s-1)\varepsilon, \quad x \ge 1.$$

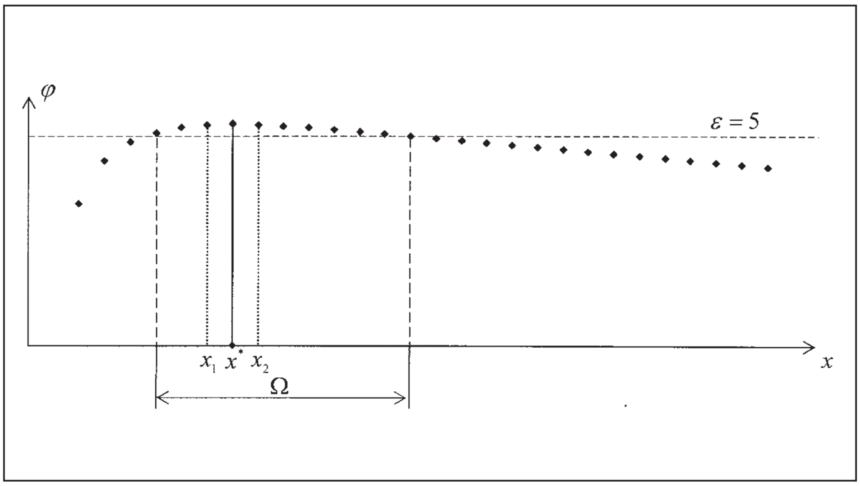

Fig. 1

According to Definition 4, an identically distributed system is optimal at the point x at which $\overline{\Delta}_\varepsilon(x)$ reaches its maximum. Let us show that this is possible at the point $x = \sqrt{\dfrac{(s-1)T^n}{\varepsilon}}$. In fact, we have

$$\overline{\Delta}'_\varepsilon(x) = \frac{(s-1)T^n}{x^2} - \varepsilon \quad \text{and} \quad \overline{\Delta}''_\varepsilon(x) = -\frac{2\,T^n\,(s-1)}{x^3} < 0 \ \text{ since } \ s \geq 2 \ \text{and} \ x > 0\,.$$

Hence, the function $\overline{\Delta}_\varepsilon(x)$ reaches its maximum at the point at which $\overline{\Delta}'_\varepsilon(x) = 0$, i.e., we have $x^* = \sqrt{\dfrac{(s-1)T^n}{\varepsilon}}$. The integer points at which the maximum of the function $\overline{\Delta}_\varepsilon(x)$ is reached are $n = [x^*]$ or $n = [x^*] + 1$. Thus, any of the numbers $\left[\sqrt{\dfrac{(s-1)T^n}{\varepsilon}}\right]$ or $\left[\sqrt{\dfrac{(s-1)T^n}{\varepsilon}}\right] + 1$ for which the function $\overline{\Delta}_\varepsilon(x)$ assumes its maximum can be chosen in the capacity of n.

The sufficiency follows from the properties of convexity of the function $\overline{\Delta}_\varepsilon(x)$ on the segment $[2, n]$ when $s \leq p$.

3. CRITERIA OF EFFICIENCY OF SYSTEMS OF IDENTICALLY DISTRIBUTED COMPETING PROCESSES WHEN THE NUMBER OF PROCESSORS IS LIMITED

Let us formulate the necessary and sufficient conditions (criteria) of efficiency and optimality of identically distributed systems of competing processes when the number of processors is limited.

It has been shown in Sec. 1 that, taking into account the parameter $\varepsilon > 0$ that characterizes the time of additional system expenditures for the organization of parallel use of blocks by a set of distributed competing processes, for computation of the minimal total time $T(p, n, s, \varepsilon)$ in the asynchronous and second synchronous modes, the following formulas take place for the class of identically distributed competing processes:

$$T(p,n,s,\varepsilon) = \begin{cases} k\,T^n + (p-1)t^n_{\max} & \text{when } s = kp,\ k > 1,\ T^n > pt^n_{\max}; \\ (k+1)T^n + (r-1)t^n_{\max} & \text{when } s = kp + r,\ k \geq 1,\ 1 \leq r < p,\ T^n > pt^n_{\max}, \end{cases}$$

where $T^n = \sum\limits_{i=1}^{n} t'_i$ is the total run time of execution of each block Q_j with the help of n processes and $t^n_{\max} = \max\limits_{1 \leq i \leq n} t'_i$.

The theorem formulated below is true.

THEOREM 6. If we have $p, n, s \geq 3$, $n = s \neq 3$, and $\varepsilon \leq \min\limits_{1 \leq i \leq n} t'_i$, then an identically distributed system consisting of n competing processes in a multiprocessor system with p processors is efficient if and only if the following conditions hold:

$$sn \geq \begin{cases} 2(kn + p - 1) & \text{if } s = kp, \ k > 1; \\ 2((k+1)n + r - 1) & \text{if } s = kp + r, \ k \geq 1, \ 1 \leq r < p. \end{cases}$$

Proof. 1. In the case when $s = kp$ and $k > 1$, the efficiency condition is equivalent to the inequality

$$\frac{(s-k)T^n - (p-1)t_{\max}^n}{\varepsilon} \geq kn + p - 1.$$

According to the condition $\varepsilon \leq \min\limits_{1 \leq i \leq n} t'_i$ of the theorem, the following inequality is true:

$$\frac{(s-k)T^n - (p-1)t_{\max}^n}{\varepsilon} \geq (s-k)n - p + 1.$$

These inequalities imply that $sn \geq 2(kn + p - 1)$.

2. When $s = kp + r$, $k \geq 1$, and $1 \leq r < p$, the following inequalities hold true:

$$\frac{(s-k-1)T^n - (r-1)t_{\max}^n}{\varepsilon} \geq (k+1)n + r - 1,$$

$$\frac{(s-k-1)T^n - (r-1)t_{\max}^n}{\varepsilon} \geq (s-k-1)n - r + 1,$$

whence $sn \geq 2((k+1)n + r - 1)$.

Let us formulate and prove the criteria of existence of an efficient system of identically distributed competing processes, depending on overheads $\varepsilon > 0$, when the number of processors is limited.

THEOREM 7. In order that an identically distributed efficient system of competing processes exist for given $p \geq 2$, T^n, and $\varepsilon > 0$, it is necessary and sufficient that the following conditions be true.

1. When $s = kp$, we have $k > 1$,

$$\varepsilon \leq \begin{cases} \varphi_1\left(\dfrac{1+\sqrt{p}}{k}\right) & \text{if } \dfrac{1+\sqrt{p}}{k} \text{ is integer;} \\ \max\left\{\varphi_1\left(\left[\dfrac{1+\sqrt{p}}{k}\right]\right), \varphi_1\left(\left[\dfrac{1+\sqrt{p}}{k}\right]+1\right)\right\} & \text{if } \dfrac{1+\sqrt{p}}{k} \text{ is not integer,} \end{cases}$$

where $\varphi_1(x) = (p-1)T^n(kx-1)/x(kx+p-1)$ and $[x]$ is the greatest integer that does not exceed x.

2. When $s = kp + r$, $k \geq 1$, and $1 \leq r < p$, we have

$$\varepsilon \leq \begin{cases} \varphi_2(x) & \text{if } x \text{ is integer;} \\ \max\{\varphi_2([x]), \varphi_2([x]+1)\} & \text{if } x \text{ is not integer,} \end{cases}$$

where $\varphi_2(x) = \dfrac{[(p-1)kx + (r-1)(x-1)]T^n}{x[(k+1)x + r - 1]}$ and $[x]$ is the greatest integer that does not exceed x, $x = \dfrac{r-1}{(p-1)k + r - 1} \times \left(1 + \sqrt{1 + \dfrac{(p-1)k + r - 1}{k+1}}\right)$.

Proof. By virtue of formulas (4), when $s = kp$ and > 1, the efficiency condition is of the form $\overline{\Delta}_\varepsilon(n) = (p-1)(kT^n - t) - (kn + p - 1)\varepsilon \ge 0$, which is equivalent to the inequality $\varepsilon \le \dfrac{(p-1)T^n(kn-1)}{n(kn+p-1)}$.

The function $\varphi_1(x) = (p-1)T^n(kx-1)/x(kx+p-1)$, $x > 0$, reaches its maximum when $x = \dfrac{1+\sqrt{p}}{k}$. Hence, when $s = kp$, $k > 1$, for an identically distributed efficient system, the number of processes must be $n = x = \dfrac{1+\sqrt{p}}{k}$ if $\dfrac{1+\sqrt{p}}{k}$ is integer and $n = x \in \left\{\left[\dfrac{1+\sqrt{p}}{k}\right], \left[\dfrac{1+\sqrt{p}}{k}\right] + 1\right\}$ if $\dfrac{1+\sqrt{p}}{k}$ is not integer.

When $s = kp + r$, $k \ge 1$, $1 \le r < p$, the efficiency condition is of the form

$$\varepsilon \le \frac{[(p-1)kn + (r-1)(n-1)]T^n}{n[(k+1)n + r - 1]}.$$

After introducing the function $\varphi_2(x) = [(p-1)kx + (r-1)(x-1)]T^n / x[(k+1)x + r - 1]$, $x > 0$, it may be shown that it reaches its maximum when

$$x = \frac{r-1}{(p-1)k + r - 1}\left(1 + \sqrt{1 + \frac{(p-1)k + r - 1}{k+1}}\right).$$

Consequently, an identically distributed system is efficient if the number of its processes equals $n = x$ when x is integer and $n = x \in \{[x], [x]+1\}$ when x is not integer.

The proof of sufficiency is similar to that used in Theorem 4.

The obtained criteria of efficiency and optimality of systems of identically distributed competing processes can be used in designing system and application software for MSs and networks during the solution of problems of optimal use of computational resources.

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